

A Burgers-KPZ Type Parabolic Equation for the Path-Independence of the Density of the Girsanov Transformation

Aubrey Truman^b, Feng-Yu Wang^{a,b}, Jiang-Lun Wu^b and Wei Yang^b

^aSchool of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

^bDepartment of Mathematics, Swansea University, Singleton Park, Swansea, SA2 8PP,
and Wales Institute of Mathematical and Computational Sciences, UK

E-mail: a.truman@swansea.ac.uk, f.y.wang@swansea.ac.uk,
j.l.wu@swansea.ac.uk, mawy@swansea.ac.uk

Abstract

Let X_t solve the multidimensional Itô's stochastic differential equations on \mathbb{R}^d

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

where $b : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is smooth in its two arguments, $\sigma : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is smooth with $\sigma(t, x)$ being invertible for all $(t, x) \in [0, \infty) \times \mathbb{R}^d$, B_t is d -dimensional Brownian motion. It is shown that, associated to a Girsanov transformation, the stochastic process

$$\int_0^t \langle (\sigma^{-1}b)(s, X_s), dB_s \rangle + \frac{1}{2} \int_0^t |\sigma^{-1}b|^2(s, X_s)ds$$

is a function of the arguments t and X_t (i.e., path-independent) if and only if $b = \sigma \sigma^* \nabla v$ for some scalar function $v : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying the time-reversed KPZ type equation

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2} \left[(Tr(\sigma \sigma^* \nabla^2 v))(t, x) + |\sigma^* \nabla v|^2(t, x) \right].$$

The assertion also holds on a connected complete differential manifold.

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1 Introduction and motivations

The present paper is mainly concerned with a link of Itô's stochastic differential equations (SDEs) to nonlinear parabolic partial differential equations (PDEs) of Burgers-KPZ type, by virtue of Girsanov transformation. Our result presents a characterization of the path-independence property for the density process of Girsanov transformation to SDEs. Our second interest is to establish such a connection between SDEs and nonlinear PDEs on complete differential manifolds.

Since the pioneering work of J.M. Burgers in 1930s (cf. e.g. [3]), Burgers equation – the simplest nonlinear PDE

$$\frac{\partial}{\partial t}u(t, x) + \lambda u(t, x) \frac{\partial}{\partial x}u(t, x) = \nu \frac{\partial^2}{\partial x^2}u(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}$$

has received a great attention both in mathematics and physics. Wherein the parameter $\lambda \in \mathbb{R}$ measures the strength of the nonlinearity, $\nu > 0$ stands for the viscosity and the (linear) viscous dissipation term on the right hand side of the equation is for the sake of softening shock wave phenomena.

Fix $d \in \mathbb{N}$, let \mathbb{R}^d be the d -dimensional Euclidean space with the inner product being denoted by $\langle \cdot, \cdot \rangle$. The multidimensional analogue to the above Burgers equation is the so called higher dimensional Burgers equation for a vorticity-free velocity field $\mathbf{u} : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ (cf. e.g., [1]) which reads the following

$$\frac{\partial}{\partial t}\mathbf{u} + \lambda(\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u}$$

where ∇ stands for the space gradient, the dot product $\mathbf{u} \cdot \nabla := \langle \mathbf{u}, \nabla \rangle$, and $\Delta := \nabla \cdot \nabla$ the Laplace operator on \mathbb{R}^d . Nowadays, Burgers equation is significant in the mathematical modeling of the large scale structure of the universe with complexity. The equation appears in many fields like aerodynamics, fluid dynamics (in particular, hydrodynamics), polymers and disordered systems, turbulence and propagation of chaos, as well as in shock wave and conservation laws – to name just a few. Among many interesting and important investigations, a breakthrough study has been made by three physicists M.P. Kardar, G. Parisi, and Y.-C. Zhang ([21]) for modeling the time evolution of the profile of a growing interface with the name of Kardar-Parisi-Zhang equation, or in short, KPZ equation. The KPZ equation describes the macroscopic properties of a wide variety of growth processes, such as growth by ballistic deposition and the Eden model (cf. [22]). For a

more mathematical account of the KPZ equation, the reader is referred to [15]. The link of the KPZ equation to multidimensional Burgers equation can be explicated as follows. It is a natural assumption that the field \mathbf{u} is often generated by a potential function (i.e., the profile) $u : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$\mathbf{u}(t, \cdot) = -\nabla u(t, \cdot), \quad t \in [0, \infty)$$

which, from the multidimensional Burgers equation, gives to the following KPZ equation for u

$$\frac{\partial}{\partial t} u(t, x) = \nu \Delta u(t, x) + \frac{\lambda}{2} |\nabla u(t, x)|^2.$$

Clearly, the above KPZ equation describes the large-distance, long-time dynamics of the growth process specified by a single-valued height $u(t, x)$ on a substrate $x \in \mathbb{R}^d$. It reflects the competition between the surface tension smoothing forces $\nu \Delta u(t, x)$ and $\frac{\lambda}{2} |\nabla u(t, x)|^2$ (the nonlinear term of u represents the tendency for growth to occur preferentially in the local normal direction to the surface).

When the diffusion coefficient $\sigma \equiv \sigma_0$, a constant, very interesting and new links of (stochastic) multidimensional Burgers' equations to (stochastic) Hamilton-Jacobi-Bellman (in short, HJB) equations and the continuity equation have been thoroughly investigated by Truman and Zhao in [35, 36, 37] (see also the early works [12, 34] for a bridge between the diffusion equations and the Schrödinger equation, now called the Elworthy-Truman formula). In this content, Hamilton-Jacobi continuity equations provide the key to obtaining asymptotic expansions in ascending powers of σ_0 for solutions of the corresponding heat (and Schrödinger) wave functions in this setting. Actually, the iterated Hamilton-Jacobi-continuity equations derived there inspired our consideration carried out in the present work.

Nowadays, because of their ubiquity, Burgers equation, the KPZ equation, and the HJB equations (as well as any of their advances studies) maintain a very hot research topic on both theoretical and applied aspects in various fields involving disordered systems and non-equilibrium dynamics. The applied aspect links to many diverse areas ranging from physics, biochemistry, and climate and ocean studies (cf. e.g. [30, 38]), to economical and financial studies (cf. [16, 17, 31, 4, 39]). There are many works in the literature devoted to analytic aspect of the equations themselves as well as to computational aspect (cf. e.g. [6, 13, 24, 25, 29, 5] and references therein).

On the other hand, the theory of SDEs has been very well developed since the seminal work of the great Japanese mathematician Kiyosi Itô in the mid 1940s, cf. [20]. Since then, SDEs have profound impacts on differential geometry and PDEs (cf. [10, 11, 19, 26] and most recently [32]).

In recent years, due to the necessity of introducing stochastic volatility as the measurement of uncertainty in modeling of financial markets, stochastic differential equations receive a huge attention from both theoretical and practical aspects, cf. e.g. [14, 19, 23, 27, 28, 33]. The primary point here is to model the price dynamics or the wealth growth by utilizing SDEs, after established a so-called real world probability space (cf. e.g. the seminal paper [2] by Black and Scholes). It is a pivotal problem to characterize the path-independence property for certain utility functions in an equilibrium market. We will give some concrete exposition of this point after the presentation of our first main result, which is the motivation of our study from economics and finance.

The object of the present paper is to explore a novel link from Itô's SDEs to nonlinear parabolic PDEs of Burgers-KPZ type with our particular attention to derive such a connection from SDEs to nonlinear PDEs on differential manifolds. Our results give a characterization of path-independence of the density of the Girsanov transformation for SDEs in terms of a nonlinear parabolic PDE of Burgers-KPZ type.

The rest of the paper is organized as follows. In the next section, we first give a brief account of the Girsanov transformation for multidimensional SDEs on \mathbb{R}^d , then we formulate our result on the characterization of path-independence of the Girsanov density and we give some further account of our result to relevant studies on path-independence features in economics and finance. We then present our proof to the theorem and we end Section 2 with an exposition of the one dimension case. Section 3, the final section, is devoted to the extension of the connection of SDEs to nonlinear equation of Burgers-KPZ type on connected complete differential manifolds.

2 The characterization theorem on \mathbb{R}^d

2.1 Preliminaries on SDEs and the Girsanov transformation

Let us start with the general framework of stochastic differential equations by following [19]. Given a complete probability space (Ω, \mathcal{F}, P) with a usual filtration $\{\mathcal{F}_t\}_{t \in [0, \infty)}$. Let $C([0, \infty), \mathbb{R}^d)$ be the space of all \mathbb{R}^d -valued, continuous functions defined on $[0, \infty)$. It is known that $C([0, \infty), \mathbb{R}^d)$ is a complete, separable metric space under the metric

$$\rho(u, v) := \sum_{k=1}^{\infty} 2^{-k} (\max_{0 \leq t \leq k} |u(t) - v(t)| \wedge 1), \quad u, v \in C([0, \infty), \mathbb{R}^d).$$

We endow the topological σ -algebra $\mathcal{B}(C([0, \infty), \mathbb{R}^d))$ on $C([0, \infty), \mathbb{R}^d)$ so that $(C([0, \infty), \mathbb{R}^d), \mathcal{B}(C([0, \infty), \mathbb{R}^d)))$ forms a measurable space, and further we denote by $\mathcal{B}_t(C([0, \infty), \mathbb{R}^d))$ the sub- σ -algebra of $\mathcal{B}(C([0, \infty), \mathbb{R}^d))$ generated by the family $\{C([0, \infty), \mathbb{R}^d) \ni u \mapsto u(s) : 0 \leq s \leq t\}$ for $t \in [0, \infty)$. As usual, $\mathbb{R}^d \otimes \mathbb{R}^m$ (with $m \in \mathbb{N}$) stands for the totality of real $d \times m$ matrices (realised alternatively by identifying $\mathbb{R}^d \otimes \mathbb{R}^m$ with dm -dimensional Euclidean space) endowed with the Hilbert-Schmidt norm

$$|a| = \sqrt{\sum_{j=1}^d \sum_{k=1}^m |a_k^j|^2}, \quad a = (a_k^j)_{d \times m} \in \mathbb{R}^d \otimes \mathbb{R}^m$$

and $\mathcal{B}(\mathbb{R}^d \otimes \mathbb{R}^m)$ denotes the topological σ -algebra on $\mathbb{R}^d \otimes \mathbb{R}^m$. Moreover, we use the notation $\mathcal{A}^{d,m}$ to denote the collection of all $\mathcal{B}([0, \infty)) \times \mathcal{B}(C([0, \infty), \mathbb{R}^d)) / \mathcal{B}(\mathbb{R}^d \otimes \mathbb{R}^m)$ -measurable mappings

$$a : [0, \infty) \times C([0, \infty), \mathbb{R}^d) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$$

such that for each $t \in [0, \infty)$, the mapping

$$u \in C([0, \infty), \mathbb{R}^d) \mapsto a(t, u) \in \mathbb{R}^d \otimes \mathbb{R}^m$$

is $\mathcal{B}_t(C([0, \infty), \mathbb{R}^d)) / \mathcal{B}(\mathbb{R}^d \otimes \mathbb{R}^m)$ -measurable.

Given $b \in \mathcal{A}^{d,1}$ and $\sigma \in \mathcal{A}^{d,d}$, we consider the following stochastic differential equation of the Markovian type for a d -dimensional continuous process $X = (X_t)_{t \in [0, \infty)}$

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0 \quad (2.1)$$

where

$$X_t = \begin{bmatrix} X_t^1 \\ X_t^2 \\ \vdots \\ X_t^d \end{bmatrix}, \quad b = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^d \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_1^1 & \cdots & \sigma_d^1 \\ \sigma_1^2 & \cdots & \sigma_d^2 \\ \vdots & & \vdots \\ \sigma_1^d & \cdots & \sigma_d^d \end{bmatrix}, \quad B_t = \begin{bmatrix} B_t^1 \\ B_t^2 \\ \vdots \\ B_t^d \end{bmatrix},$$

so equation (2.1) in terms of its components is

$$dX_t^j = b^j(t, X_t)dt + \sum_{k=1}^d \sigma_k^j(t, X_t)dB_t^k, \quad j = 1, 2, \dots, d \quad (2.2)$$

where σ_k^j stands for the (j, k) -entry of the $d \times d$ -matrix σ , for $j, k = 1, 2, \dots, d$, and $B_t = (B_t^1, B_t^2, \dots, B_t^d)^*$ is an d -dimensional $\{\mathcal{F}_t\}_{t \in [0, \infty)}$ -Brownian motion. It is well known, from e.g. [19] (cf. Theorem IV.3.1), that under the usual conditions of linear growth and locally Lipschitz, to be precise, the coefficients $b : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ satisfy linear growth and locally Lipschitz condition, C^1 with respect to the first variable, and C^2 with respect to the second variable, there exists a unique solution to equation (2.1) with given initial data X_0 . By Stroock-Varadhan's martingale problem [33], X_t is associated with the following second order elliptic differential operator (called the Markov generator)

$$L_t f(x) = \frac{1}{2} \sum_{i,j=1}^d a^{ij}(t, x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{j=1}^d b^j(t, x) \frac{\partial f(x)}{\partial x_j}, \quad f \in C^2(\mathbb{R}^d)$$

with $a(t, x) := \sigma(t, x)\sigma^*(t, x)$, where $\sigma^*(t, x)$ stands for the transposed matrix of $\sigma(t, x)$. In component form, $a^{ij}(t, x) := \sum_{k=1}^d \sigma_k^i(t, x)\sigma_k^j(t, x)$.

The celebrated Girsanov theorem provides a very powerful probabilistic tool to solve equation (2.1) under the name of the *Girsanov transformation* or the *transformation of the drift*. Let $\gamma \in \mathcal{A}^{d,1}$ satisfy the following condition

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds \right) \right] < \infty, \quad \forall t > 0.$$

Then, by Girsanov theorem (cf e.g. Theorem IV 4.1 of [19]),

$$\exp \left(\int_0^t \gamma(s, X_s) dB_s - \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds \right), \quad t \in [0, \infty)$$

is an $\{\mathcal{F}_t\}$ -martingale. Furthermore, for $t \geq 0$, we define

$$Q_t := \exp \left(\int_0^t \gamma(s, X_s) dB_s - \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds \right) \cdot P$$

or equivalently in terms of the Radon-Nikodym derivative

$$\frac{dQ_t}{dP} = \exp \left(\int_0^t \gamma(s, X_s) dB_s - \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds \right).$$

Then, for any $T > 0$,

$$\tilde{B}_t := B_t - \int_0^t \gamma(s, X_s) ds, \quad 0 \leq t \leq T$$

is an $\{\mathcal{F}_t\}$ -Brownian motion under the probability Q_T . Moreover, X_t satisfies

$$dX_t = [b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t)]dt + \sigma(t, X_t)d\tilde{B}_t, \quad t \geq 0.$$

One can then discuss comprehensively the *existence and uniqueness as well as the structure of solutions* to the initial value problem for equation (2.1) by appealing the above argument with suitable choice of γ . Here we want to explore such transformation to another link to partial differential equations.

2.2 The characterization theorem and its link to economics and finance studies

From now on in the paper, we assume the coefficient σ satisfies that the matrix $\sigma(t, x)$ is invertible, for any $(t, x) \in [0, \infty) \times \mathbb{R}^d$ (and consequently so is the symmetric matrix $a(t, x) = \sigma(t, x)\sigma^*(t, x)$). Moreover, we specify the above function γ by

$$\gamma(t, x) = -(\sigma(t, x))^{-1}b(t, x)$$

so that $b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t) = 0$, and hence we further require b and σ satisfy

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^t |(\sigma(s, X_s))^{-1}b(s, X_s)|^2 ds \right) \right] < \infty, \quad \forall t > 0.$$

Thus the associated probability measure Q_t is determined by

$$\begin{aligned} \frac{dQ_t}{dP} = & \exp \left(- \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle \right. \\ & \left. - \frac{1}{2} \int_0^t |(\sigma(s, X_s))^{-1} b(s, X_s)|^2 ds \right). \end{aligned}$$

Set

$$\hat{Z}_t := - \ln \frac{dQ_t}{dP}$$

that is

$$\hat{Z}_t = \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle + \frac{1}{2} \int_0^t |(\sigma(s, X_s))^{-1} b(s, X_s)|^2 ds.$$

Clearly, \hat{Z}_t is a one dimensional stochastic process with the stochastic differential form

$$d\hat{Z}_t = \frac{1}{2} |(\sigma(t, X_t))^{-1} b(t, X_t)|^2 dt + \langle (\sigma(t, X_t))^{-1} b(t, X_t), dB_t \rangle.$$

We are now ready to state the first main result of this paper. It gives a necessary and sufficient condition, and hence a characterization of path-independence of the density \hat{Z}_t of the Girsanov transformation for SDEs in terms of a nonlinear parabolic PDE of Burgers-KPZ type. Namely we establish a bridge from SDE (2.1) to a nonlinear parabolic PDE of Burgers-KPZ type.

Theorem 2.1 *Let $v : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a scalar function which is C^1 with respect to the first variable and C^2 with respect to the second variable. Then*

$$\begin{aligned} v(t, X_t) = & v(0, X_0) + \frac{1}{2} \int_0^t |(\sigma(s, X_s))^{-1} b(s, X_s)|^2 ds \\ & + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle \end{aligned} \quad (2.3)$$

equivalently,

$$\frac{dQ_t}{dP} = \exp\{v(0, X_0) - v(t, X_t)\}, \quad t \in [0, \infty)$$

holds if and only if

$$b(t, x) = (\sigma \sigma^* \nabla v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d \quad (2.4)$$

and v satisfies the following time-reversed KPZ type equation

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2} \{ [Tr(\sigma \sigma^* \nabla^2 v)](t, x) + |\sigma^* \nabla v|^2(t, x) \} \quad (2.5)$$

where $\nabla^2 v$ stands for the Hessian matrix of v with respect to the second variable.

Remark 2.2 *The derived time-reversed KPZ type equation (2.5) is contained as a special version of the stochastic HJB equation derived in [37]. It is an interesting question to see if one can recover a fuller picture of the mathematical physics of the stochastic HJB equations in [35, 36, 37] by exploiting the argument developed in this paper. We will consider this problem in our future work.*

Before presenting the proof to Theorem 2.1, we would like to give some links of our result to economic and financial studies.

Remark 2.3 *Recall that in economics and finance studies, a conventional kind of equilibrium financial market can be characterized by the utility function of a representative agent (see e.g., [4, 7, 8, 9]). Given the probability measure P as an objective probability in the market model, one can interpret our process X_t as the wealth (or the assets price) of the representative agent in a multi-assets market. If the class of utility functions is one of differentiable, increasing, and strictly concave time-separable von Neumann-Morgenstern utility functions, then the representative agent maximizes his/her expected utility U . Cox and Leland in [4] show that the path-independence property is necessary for expected utility maximization. By path-independence, they mean that the value of a portfolio will depend only on the assets price at that time, not on the path followed by the assets in reaching that price (vector). Namely, the utility function U depends on the state price X_t and time t , for $t \geq 0$, that is, the function U is of the form $U(X_t, t)$. On the other hand, Dybvig and Ross in [9] show that, in an equilibrium market, the marginal utility in each state is proportional to a consistent state-price density function. In addition, in an equilibrium market, there must exist a risk neutral*

probability measure Q which is absolutely continuous with respect to P . The Radon-Nikodym derivative $Z = \frac{dQ}{dP}$ gives the state-price density [16]. Combining the above $U(X_t, t)$, therefore, the Radon-Nikodym derivative is also in the form of

$$Z(X_t, t) = \frac{dQ_t}{dP}.$$

Clearly, our Theorem 2.1 presents a necessary and sufficient condition for the above Radon-Nikodym derivative, hence a characterization for the path-independence property of the utility function.

2.3 Proof of Theorem 2.1

We start with the *necessity*. Namely, assume that there exists a scalar function $v : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ which is C^1 with respect to the first variable and C^2 with respect to the second variable such that (2.3) holds. Then by (2.3), we have

$$dv(t, X_t) = \frac{1}{2} |(\sigma(t, X_t))^{-1} b(t, X_t)|^2 dt + \langle (\sigma(t, X_t))^{-1} b(t, X_t), dB_t \rangle. \quad (2.6)$$

Now by viewing $v(t, X_t)$ as the composition of the deterministic $C^{1,2}$ -function $v : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ with the continuous semi-martingale X_t , we can apply Itô's formula to $v(t, X_t)$ and further with the help of equation (2.1), we have the following derivation

$$\begin{aligned} dv(t, X_t) = & \left\{ \frac{\partial}{\partial t} v(t, X_t) + \frac{1}{2} [Tr(\sigma \sigma^*) \nabla^2 v](t, X_t) \right. \\ & \left. + \langle b, \nabla v \rangle(t, X_t) \right\} dt + \langle (\sigma^* \nabla v)(t, X_t), dB_t \rangle \end{aligned} \quad (2.7)$$

since

$$\langle \nabla v(t, X_t), \sigma(t, X_t) dB_t \rangle = \langle \sigma^*(t, X_t) \nabla v(t, X_t), dB_t \rangle.$$

Now comparing (2.6) and (2.7) and using the uniqueness of Doob-Meyer's decomposition of continuous semi-martingale, we conclude that the coefficients of dt and dB_t must coincide, respectively, namely

$$(\sigma^{-1} b)(t, X_t) = (\sigma^* \nabla v)(t, X_t)$$

and

$$\frac{1}{2}|(\sigma^{-1}b)(t, X_t)|^2 = \frac{\partial}{\partial t}v(t, X_t) + \frac{1}{2}[Tr(\sigma\sigma^*\nabla^2v)](t, X_t) + \langle b, \nabla v \rangle(t, X_t)$$

holds for all $t > 0$. Since the SDE (2.1) is non-degenerate, the support of $X_t, t \in [0, \infty)$ is the whole space \mathbb{R}^d . Hence, the following two equalities

$$(\sigma^{-1}b)(t, x) = (\sigma^*\nabla)v(t, x) \quad (2.8)$$

and

$$\frac{1}{2}|(\sigma^{-1}b)(t, x)|^2 = \frac{\partial}{\partial t}v(t, x) + \langle b, \nabla v \rangle(t, x) + \frac{1}{2}[Tr(\sigma\sigma^*\nabla^2v)](t, x) \quad (2.9)$$

hold on $[0, \infty) \times \mathbb{R}^d$. It is clear that equality (2.8) is nothing but equality (2.4), while by (2.4) the equality (2.9) reduces to equation (2.5).

Now let us turn to the *sufficiency*. We assume that there exists a $C^{1,2}$ scalar function $v : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ solving equation (2.5). We specify the drift b of SDE (2.1) via (2.4), namely

$$b(t, x) = (\sigma\sigma^*\nabla v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Combining equality (2.4) and equation (2.5) with equality (2.7), we have

$$\begin{aligned} dv(t, X_t) &= \left[-\frac{1}{2}|\sigma^*\nabla v|^2(t, X_t) + \langle b, \nabla v \rangle(t, X_t) \right] dt \\ &\quad + \langle (\sigma^*\nabla v)(t, X_t), dB_t \rangle \\ &= \frac{1}{2}|\sigma^{-1}b|^2(t, X_t)dt + \langle (\sigma^{-1}b)(t, X_t), dB_t \rangle. \end{aligned}$$

This clearly implies equality (2.3) by taking stochastic integration. We are done. *Q.E.D.*

2.4 The special case of $d = 1$

In this subsection, we would like to discuss our Theorem 2.1 on \mathbb{R} — the simplest case. We start with SDE in one dimension:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0 \quad (2.10)$$

with the diffusion coefficient satisfies that $\sigma(t, x) \neq 0$ for all $(t, x) \in [0, \infty) \times \mathbb{R}$. In this case we have

$$\gamma(t, x) = -\frac{b(t, x)}{\sigma(t, x)}.$$

We set

$$u(t, x) := \frac{b(t, x)}{\sigma^2(t, x)} = -\frac{\gamma(t, x)}{\sigma(t, x)}, \quad (t, x) \in [0, \infty) \times \mathbb{R}. \quad (2.11)$$

With the assumption on γ for the Girsanov theorem, we can rephrase our Theorem 2.1 in a slightly more concise manner

Theorem 2.4 *Let $v : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be C^1 with respect to the first variable and C^2 with respect to the second variable. Then*

$$v(t, X_t) = \frac{dQ_t}{dP} = \exp \left(v(0, X_0) - \int_0^t \frac{b(s, X_s)}{\sigma(s, X_s)} dB_s - \frac{1}{2} \int_0^t \left| \frac{b(s, X_s)}{\sigma(s, X_s)} \right|^2 ds \right) \quad (2.12)$$

if and only if there exists a C^1 -function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$b(t, X_t) = \Phi(u(t, X_t)), \quad \forall t \geq 0$$

and the function $u(t, x)$ satisfies the following generalized Burgers equation (again time-reversed)

$$\frac{\partial}{\partial t} u(t, x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi_1(u(t, x)) - \frac{1}{2} \frac{\partial}{\partial x} \Psi_2(u(t, x)) \quad (2.13)$$

where

$$\Psi_1(r) := \int \frac{\Phi(r)}{r} dr, \quad \Psi_2(r) := r\Phi(r), \quad r \in \mathbb{R}.$$

Proof The proof is in the same manner as the proof to Theorem 2.1 together with the combination of the introduction and properties of the functions Ψ_1 and Ψ_2 . We omit the whole derivation here. *Q.E.D.*

Remark 2.5 *We would like to point out that the function u defined in formula (2.11) has the following explanation. Actually, from Itô formula, one may see that the square of Brownian motion has certain contribution to the drift of the stochastic differential equation. So the composition $u(t, X_t)$ of the function u with the process X_t may characterize the proportion of the drift part with respect to the diffusion part in equation (2.10). Surprisingly, this function u satisfies the nonlinear parabolic PDE of Burgers type (2.13).*

Our PDE (2.13) covers much more classes of specific nonlinear PDEs. Now let us give several examples to explicate this point.

Example 2.6 Give a constant $\sigma > 0$. Let $b(t, x) = \sigma^2 u(t, x)$ and $\sigma(t, x) \equiv \sigma$, our SDE (2.1) then becomes

$$dX_t = \sigma^2 u(t, X_t) dt + \sigma dB_t.$$

The C^1 -function Φ is simply given by $\Phi(r) = \sigma^2 r$ and the corresponding PDE (2.13) is a classical Burgers equation (time-reversed)

$$\frac{\partial}{\partial t} u(t, x) = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} u(t, x) - \sigma^2 u(t, x) \frac{\partial}{\partial x} u(t, x).$$

This example recovers the main result obtained in Hodges and Carverhill [16]. Moreover, our Theorem 2.4 also covers the results obtained in Hodges and Liao [17] and in Stein and Stein [31].

The next example shows that our PDE (2.13) can be a porous media type partial differential equation.

Example 2.7 We fix $m \in \mathbb{N}$. Let $a(t, x) = m[u(t, x)]^m$ and $b(t, x) = \sqrt{m}[u(t, x)]^{\frac{m-1}{2}}$, our SDE (2.10) then becomes

$$dX_t = m[u(t, X_t)]^m dt + \sqrt{m}[u(t, X_t)]^{\frac{m-1}{2}} dB_t.$$

The C^1 -function Φ is then given by $\Phi(r) = mr^m$ and the corresponding PDE (2.13) is a porous media type nonlinear PDE

$$\frac{\partial}{\partial t} u(x, t) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} u^m(t, x) - m \frac{\partial}{\partial x} u^{m+1}(t, x).$$

Our third example is to show that in the time-homogeneous case in the sense that b and σ are functions of the variable $x \in \mathbb{R}$ only, the corresponding PDE (2.13) then determines a harmonic function.

Example 2.8 Let $b(t, x) = b(x)$ and $\sigma(t, x) = \sigma(x)$, our SDE (2.10) then reads as follows

$$dX_t = b(X_t) dt + \sigma(X_t) dB_t$$

and the corresponding PDE (2.13) is a second order elliptic equation for harmonic functions

$$\frac{\partial^2}{\partial x^2} \Psi_1(u(x)) + \frac{\partial}{\partial x} \Psi_2(u(x)) = 0$$

where

$$\Psi_1(r) = \int \frac{\Phi(r)}{r} dr, \quad \Psi_2(r) = r\Phi(r), \quad r \in \mathbb{R}.$$

3 Extension to differential manifolds

In this final section, we extend our Theorem 2.1 for SDEs on a general connected complete differential manifold. We start with the following observation. In the situation of SDE (2.1) on \mathbb{R}^d , let $g_t = (g_t^{ij}(\cdot)) := (\sigma\sigma^*)^{-1}(t, \cdot)$. Then we have a time-dependent metric on \mathbb{R}^d defined as follow

$$\langle x, y \rangle_{g_t} := \sum_{i,j=1}^d g_t^{ij} x_i y_j = \langle g_t x, y \rangle, \quad x, y \in \mathbb{R}^d.$$

Let ∇_{g_t} and Δ_{g_t} be the associated gradient and Laplacian, respectively. Then the generator for the solution to SDE (2.1) can be reformulated as follows (cf. [19])

$$L_t f = \frac{1}{2} \Delta_{g_t} f + \langle \tilde{b}(t, \cdot), \nabla_{g_t} f \rangle_{g_t}$$

for some smooth function $\tilde{b} : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. From this point of view, we intend to extend our Theorem 2.1 to a general connected complete differential manifold.

Now let M be a d -dimensional connected complete differential manifold with a family of Riemannian metrics $\{g_t\}_{t \in [0, \infty)}$, which is smooth in $t \in [0, \infty)$. Clearly (M, g_t) is a Riemannian manifold for each $t \in [0, \infty)$. Let $\{b(t, \cdot)\}_{t \in [0, \infty)}$ be a family of smooth vector fields on M which is smooth in t as well. Let ∇_{g_t} and Δ_{g_t} denote the gradient and Laplacian operators induced by the metric g_t , respectively. Then the diffusion process on M generated by the operator

$$L_t := \frac{1}{2} \Delta_{g_t} + b(t, \cdot)$$

can be constructed by solving the following SDE on M

$$dX_t = b(t, X_t)dt + \Phi_t \circ dB_t \tag{3.1}$$

where $\{B_t\}_{t \in [0, \infty)}$ is the d -dimensional Brownian motion, $\circ d$ stands for the Stratonovich differential, and Φ_t is the horizontal lift of X_t onto the frame bundle $O_t(M)$ of the Riemannian manifold (M, g_t) , namely, Φ_t solves the following equation

$$d\Phi_t = H_{\Phi_t} \circ dX_t,$$

with $H_{\Phi_t} : T(M) \rightarrow O_t(M)$ being the horizontal lift. Here $T_t(M)$ denotes the tangent bundle of M .

The following result is an extension of our Theorem 2.1 to M .

Theorem 3.1 *Let $v : [0, \infty) \times M \rightarrow \mathbb{R}$ be C^1 with respect to the first variable and C^2 with respect to the second variable. Then*

$$v(t, X_t) = v(0, X_0) + \frac{1}{2} \int_0^t |b(s, X_s)|_{g_s}^2 ds + \int_0^t \langle (\Phi_s^{-1} b(s, X_s), \circ dB_s) \rangle_{g_s} \quad (3.2)$$

holds if and only if

$$b(t, x) = (\nabla_{g_t} v)(t, x), \quad (t, x) \in [0, \infty) \times M \quad (3.3)$$

and the following time-reversed KPZ type equation

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2} [(\Delta_{g_t} v)(t, x) + |\nabla_{g_t} v|_{g_t}^2(t, x)] \quad (3.4)$$

hold, where $|z|_{g_t}^2 := \langle z, z \rangle_{g_t}$ for any vector z on M .

Proof By (3.2), we have

$$dv(t, X_t) = \frac{1}{2} |b(t, X_t)|_{g_t}^2 dt + \langle \Phi_t^{-1} b(t, X_t), \circ dB_t \rangle_{g_t}. \quad (3.5)$$

On the other hand, by (3.1) and the Itô formula, we get

$$dv(t, X_t) = \langle \Phi_t^{-1} \nabla_{g_t} v(t, X_t), \circ dB_t \rangle_{g_t} + \left\{ \frac{1}{2} \Delta_{g_t} v + \langle b, \nabla_{g_t} v \rangle_{g_t} \right\} (t, X_t) dt. \quad (3.6)$$

Now combining (3.6) with (3.5), we arrive the following

$$\nabla_{g_t} v(t, X_t) = b(t, X_t)$$

and

$$\left\{ \frac{1}{2} \Delta_{g_t} v + \langle b, \nabla_{g_t} v \rangle_{g_t} \right\} (t, X_t) = \frac{1}{2} |b(t, X_t)|_{g_t}^2.$$

Since $\{X_t\}_{t \in [0, \infty)}$ is supported by the whole manifold, the above two equalities imply (3.3) and (3.4), respectively.

On the other hand, combining (3.3) and (3.4) with (3.6), we obtain (3.5), which implies (3.2) by stochastic integration. This completes the proof. *Q.E.D.*

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